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Evaluation of integrals involving powers of $(1-x^2)$ and two associated Legendre functions or Gegenbauer polynomials

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Abstract. We express the integral

$$\int_{-1}^{1} P_{l+\alpha}^{-\alpha}(x) P_{k+\beta}^{-\beta}(x)(1-x^2)^{-p-1}\,\mathrm{d}x$$

where l, k are non-negative integers and $\mathrm{Re}[\frac{1}{2}(\alpha+\beta)-p]>0$ for convergence as a multiple of a terminating Saalchützian hypergeometric function ${}_4F_3$ of unit argument and utilise our results to compute the following special integrals.

(i) $$\int_{-1}^{1} P_l^m(x) P_k^n(x)(1-x^2)^{-p-1}\,\mathrm{d}x$$

where l, k, m, n are non-negative integers and $l\geqslant m\geqslant 0$, $k\geqslant n\geqslant 0$, $\mathrm{Re}[\frac{1}{2}(m+n)-p]>0$.

(ii) $$\int_{-1}^{1} C_l^\alpha(x) C_k^\beta(x)(1-x^2)^{(\alpha+\beta-3)/2-p}\,\mathrm{d}x$$

where l, k are non-negative integers and $\mathrm{Re}[\frac{1}{2}(\alpha+\beta-1)-p]>0$. The symbols $P_\alpha^\beta(x)$ and $C_l^\alpha(x)$ represent the associated Legendre functions and Gegenbauer polynomials respectively. Both of the above integrals are expressed as multiples of a terminating Saalchützian hypergeometric function ${}_4F_3$ of unit argument. For the special case $m=n(\alpha=\beta)$ and integral p, the ${}_4F_3$ function in (i) and (ii) has $\min(p+1, |\frac{1}{2}l|+1)$ terms for $p\geqslant 0$ and $\min(-p, |\frac{1}{2}l|+1)$ for $p<0$. Our work generalises the known results for diagonal integrals to off-diagonal integrals of the type given in (i) and (ii).

1. Introduction

Associated Legendre functions and Gegenbauer polynomials appear frequently in many branches of theoretical physics where analytic expressions for integrals containing these and powers of $(1-x^2)$ are required. Recently Laursen and Mita (1981) found it necessary to evaluate integrals of the type

$$\int_{-1}^{1} [P_l^m(x)]^2(1-x^2)^{-p-1}\,\mathrm{d}x$$

with l, m, p integers, $l\geqslant m\geqslant 0$, $m-p>0$, and

$$\int_{-1}^{1} [C_l^\alpha(x)]^2(1-x^2)^{\alpha-(3/2)-p}\,\mathrm{d}x$$

with l, p integers and $\mathrm{Re}(\alpha-\frac{1}{2}-p)>0$, for applications to meson physics. They expressed their results in terms of a terminating Saalchützian hypergeometric function

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of unit argument (TSHFUA) ${}_4F_3$, in which the number of terms is $p+1$ for $p \geqslant 0$ and $(-p)$ for $p<0$. Following their work, Ullah (1984) used an operator form of the Taylor theorem to express the integral

$$\int_{-1}^{1} P_l^m(x) P_k^n(x)(1-x^2)^{-p-1}\,dx$$

with l, m, k, n, p integers and $l \geqslant m \geqslant 0$, $k \geqslant n \geqslant 0$, $\frac{1}{2}(m+n)-p>0$ in terms of a Saalchützian hypergeometric function ${}_4F_3$ of unit argument which in the diagonal case is terminating for $p \geqslant 0$ and not so for $p<0$ except when $l \pm m$ (and hence $k \pm n$) are even integers. Our purpose in the present paper is to evaluate the integral

$$\int_{-1}^{1} P_l^m(x) P_k^n(x)(1-x^2)^{-p-1}\,dx$$

under the conditions given above and the integral

$$\int_{-1}^{1} C_l^\alpha(x) C_k^\beta(x)(1-x^2)^{(\alpha+\beta-3)/2-p}\,dx$$

where l, k are non-negative integers, p is an integer and $\mathrm{Re}\,\frac{1}{2}(\alpha+\beta-1)-p>0$. To achieve our aim, we first evaluate a more general integral

$$I(l, \alpha; k, \beta; p)=\int_{-1}^{1} P_{l+\alpha}^{-\alpha}(x) P_{k+\beta}^{-\beta}(x)(1-x^2)^{-p-1}\,dx$$

where l and k are non-negative integers and $\mathrm{Re}[\frac{1}{2}(\alpha+\beta)-p]>0$ for convergence of the integral. It is observed that this integral vanishes unless $\frac{1}{2}(l-k)$ is an integer. For this case, we are able to express this integral as a multiple of a TSHFUA ${}_4F_3$. This hypergeometric function for the particular case of $m=n$ $(\alpha=\beta)$ and integral p has $\min(p+1, |\frac{1}{2}l|+1)$ terms for $p \geqslant 0$ and $\min(-p, |\frac{1}{2}l|+1)$ for $p<0$. Here $|\frac{1}{2}l|$ means the integral part of $\frac{1}{2}l$ and we have assumed that $k \geqslant l$. The *special integrals* we are interested in can be expressed in terms of this general integral. Indeed

$$\int_{-1}^{1} P_l^m(x) P_k^n(x)(1-x^2)^{-p-1}\,dx=(-1)^{m+n}\frac{(l+m)!(k+n)!}{(l-m)!(k-n)!} I(l-m, m; k-n, n; p)$$

and

$$\int_{-1}^{1} C_l^\alpha(x) C_k^\beta(x)(1-x^2)^{2(\alpha+\beta-3)/2-p}\,dx=\pi\frac{\Gamma(l+2\alpha)\Gamma(k+2\beta)}{l!k!\Gamma(\alpha)\Gamma(\beta)} I(l, \alpha-\tfrac{1}{2}; k, \beta-\tfrac{1}{2}; p).$$

This paper is organised as follows. In § 2 we exhibit the evaluation of the integral $I(l, \alpha; k, \beta; p)$. In § 3, we evaluate the *special integrals* in terms of the integral $I(l, \alpha; k, \beta; p)$. In § 4, we tabulate the special integrals for the special cases where $m=n$ $(\alpha=\beta)$ and the $p=1, 0, -1, -2$, and l and k not necessarily equal generalising the corresponding $l=k$ results tabulated by Laursen and Mita (1981).

2. Evaluation of the general integral

In this section, we wish to evaluate the general integral

$$I(l, \alpha; k, \beta; p)=\int_{-1}^{1} P_{l+\alpha}^{-\alpha}(x) P_{k+\beta}^{-\beta}(x)(1-x^2)^{-p-1}\,dx \tag{1}$$

where l and k are non-negative integers and we require

$$\mathrm{Re}[\tfrac{1}{2}(\alpha+\beta)-p]>0 \tag{2}$$

for convergence.

From the definition (equation (8.704) in Gradshteyn *et al* (1965)) of the associated Legendre function we have

$$P_{l+\alpha}^{-\alpha}(x)=\frac{1}{\Gamma(\alpha+1)}\left(\frac{1+x}{1-x}\right)^{-\alpha/2}{}_2F_1[-l-\alpha,\, l+\alpha+1;\, \alpha+1;\, \tfrac{1}{2}(1-x)] \tag{3}$$

which becomes on using

$${}_2F_1(\alpha,\beta;\gamma;z)=(1-z)^{\gamma-\alpha-\beta}{}_2F_1(\gamma-\alpha,\gamma-\beta;\gamma;z) \tag{4}$$

given in equation (9.131) in Gradshteyn *et al* (1965)

$$P_{l+\alpha}^{-\alpha}(x)=\frac{1}{\Gamma(\alpha+1)}2^{-\alpha}(1-x^2)^{\alpha/2}{}_2F_1[-l,\, l+2\alpha+1;\, \alpha+1;\, \tfrac{1}{2}(1-x)]. \tag{5}$$

When $0\leqslant x\leqslant 1$, $0\leqslant\frac{1}{2}(1-x)\leqslant\frac{1}{2}$, we use the quadratic transformation (equation (9.133) in Gradshteyn *et al* (1965) and (4) above) to arrive at

$$P_{l+\alpha}^{-\alpha}(x)=\frac{1}{\Gamma(\alpha+1)}2^{-\alpha}(1-x^2)^{\alpha/2}{}_2F_1[-\tfrac{1}{2}l,\, \tfrac{1}{2}(l+1)+\alpha;\, \alpha+1;\, 1-x^2] \tag{6a}$$

$$=\frac{1}{\Gamma(\alpha+1)}2^{-\alpha}(1-x^2)^{\alpha/2}x\,{}_2F_1[-\tfrac{1}{2}(l-1),\, \tfrac{1}{2}l+\alpha+1;\, \alpha+1;\, 1-x^2]. \tag{6b}$$

The above equations are valid for $0\leqslant x\leqslant 1$. When l is an even (odd) non-negative integer, we use (6*a*) ((6*b*)) to express $P_{l+\alpha}^{-\alpha}(x)$ as a finite series (x times a finite series) in powers of $(1-x^2)$. Indeed we have for $0\leqslant x\leqslant 1$

$$P_{l+\alpha}^{-\alpha}(x)=2^{-\alpha}\frac{(\frac{1}{2}l)!}{\Gamma[\frac{1}{2}(l+1)+\alpha]}\sum_t(-1)^t\frac{\Gamma[\frac{1}{2}(l+1)+\alpha+t]}{t!(\frac{1}{2}l-t)!\Gamma(\alpha+1+t)}(1-x^2)^{\alpha/2+t} \tag{7a}$$

for even non-negative l and

$$P_{l+\alpha}^{-\alpha}(x)=\frac{2^{-\alpha}[\frac{1}{2}(l-1)]!}{\Gamma[\frac{1}{2}l+\alpha+1]}\sum_t(-1)^t\frac{\Gamma(\frac{1}{2}l+\alpha+1+t)}{t![\frac{1}{2}(l-1)-t]!\Gamma(\alpha+1+t)}x(1-x^2)^{\alpha/2+t} \tag{7b}$$

for odd non-negative l. In the above and in what follows, the range of summations will be dictated by the non-negativity of the arguments of the factorials present and will not be explicitly indicated. Now from equation (8.737 (2)) in Gradshteyn *et al* (1965) for $0<x<1$ and non-negative integral l we have

$$P_{l+\alpha}^{-\alpha}(-x)=(-1)^l P_{l+\alpha}^{-\alpha}(x) \tag{8}$$

which shows that the integral $I(l,\alpha;k,\beta;p)$ is zero unless $\frac{1}{2}(l-k)$ is an even integer, i.e. when both l and k are even or odd integers. In what follows, we assume that $\frac{1}{2}(l-k)$ is indeed an even integer. Then (1) becomes

$$I(l,\alpha;k,\beta;p)=2\int_0^1 P_{l+\alpha}^{-\alpha}(x)P_{k+\beta}^{-\beta}(x)(1-x^2)^{-p-1}\,\mathrm{d}x. \tag{9}$$

When l and k are both even integers, we use the expansion in (7a) and integrate the resulting sum term by term using

$$\int_0^1 (1-x^2)^{(\alpha+\beta)/2-p-1+t_1+t_2}\,dx = \tfrac{1}{2}\sqrt{\pi}\,\frac{\Gamma[\frac{1}{2}(\alpha+\beta)-p+t_1+t_2]}{\Gamma[\frac{1}{2}(\alpha+\beta+1)-p+t_1+t_2]} \tag{10}$$

which is valid for $\mathrm{Re}[\frac{1}{2}(\alpha+\beta)-p+t_1+t_2]>0$ (this condition follows from the condition $\mathrm{Re}[\frac{1}{2}(\alpha+\beta)-p]>0$ given in (2) above), to arrive at

$$\begin{aligned} I(l,\alpha;k,\beta;p) &= \sqrt{\pi}\,2^{-(\alpha+\beta)}\frac{(\frac{1}{2}l)!(\frac{1}{2}k)!}{\Gamma[\frac{1}{2}(l+1)+\alpha]\Gamma[\frac{1}{2}(k+1)+\beta]}\sum_{t_1t_2}(-1)^{t_1+t_2} \\ &\quad\times\frac{\Gamma[\frac{1}{2}(l+1)+\alpha+t_1]\Gamma[\frac{1}{2}(k+1)+\beta+t_2]\Gamma[\frac{1}{2}(\alpha+\beta)-p+t_1+t_2]}{t_1!\,t_2!\,(\frac{1}{2}l-t_1)!\,(\frac{1}{2}k-t_2)!\,\Gamma(\alpha+1+t_1)} \\ &\quad\times\{\Gamma(\beta+1+t_2)\Gamma[\tfrac{1}{2}(\alpha+\beta+1)-p+t_1+t_2]\}^{-1}. \end{aligned} \tag{11}$$

Now we note that the t_2 summation can be expressed in terms of a TSHFUA ${}_3F_2$ which can be explicitly evaluated using equation (2.2(2)) in Bailey (1964). The remaining t_1 summation can be expressed in terms of a TSHFUA ${}_4F_3$ to give

$$\begin{aligned} I(l,\alpha;k,\beta;p) &= 2^{-(\alpha+\beta)}\frac{\Gamma[\frac{1}{2}(k+1)]\Gamma(-\frac{1}{2}k-\beta)\Gamma[\frac{1}{2}(\alpha+\beta)-p]\Gamma[\frac{1}{2}(\alpha-\beta)-p]}{\Gamma(\beta+1)\Gamma(-\beta)\Gamma(\alpha+1)\Gamma[\frac{1}{2}(-k+\alpha-\beta)-p]\Gamma[\frac{1}{2}(k+\alpha+\beta+1)-p]} \\ &\quad\times{}_4F_3\left(\begin{matrix}\frac{1}{2}(l+1)+\alpha, & \frac{1}{2}(\alpha+\beta)-p, & \frac{1}{2}(\alpha-\beta)-p & -\frac{1}{2}l \\ \alpha+1, & \frac{1}{2}(-k+\alpha-\beta)-p, & \frac{1}{2}(k+\alpha+\beta+1)-p & \end{matrix};1\right). \end{aligned} \tag{12}$$

The ${}_4F_3$ in (12) can be transformed into many different forms using equation (7.2(1)) in Bailey (1964). One of these forms manifests the symmetry $l,\alpha\leftrightarrow k,\beta$. However, we are interested in the transformation which in the language of (7.2(1)) in Bailey (1964) uses

$$z=\tfrac{1}{2}(\alpha-\beta)-p \qquad n=\tfrac{1}{2}l \qquad u=\alpha+1$$

to result in

$$\begin{aligned} I(l,\alpha;k,\beta;p) &= 2^{-(\alpha+\beta)}\frac{\Gamma[\frac{1}{2}(k+1)]\Gamma(-\frac{1}{2}k-\beta)\Gamma[\frac{1}{2}(\alpha+\beta)-p]\times\Gamma[\frac{1}{2}(\alpha-\beta-p)]\Gamma[\frac{1}{2}(l-k)]\Gamma[\frac{1}{2}(l+k+1)+\beta]}{\Gamma(\beta+1)\Gamma(-\beta)\Gamma(\alpha+1)\Gamma(-\frac{1}{2}k)\Gamma[\frac{1}{2}(k+1)+\beta]\times\Gamma[\frac{1}{2}(l-k+\alpha-\beta)-p]\Gamma[\frac{1}{2}(l+k+\alpha+\beta+1)-p]} \\ &\quad\times{}_4F_3\left(\begin{matrix}\frac{1}{2}(\alpha-\beta)+p+1, & \frac{1}{2}(\alpha-\beta)-p, & -\frac{1}{2}(l-1), & -\frac{1}{2}l \\ \frac{1}{2}(k-l)+1, & -\frac{1}{2}(l+k-1)-\beta, & \alpha+1 & \end{matrix};1\right). \end{aligned} \tag{13}$$

Since in the above l and k are both even integers, we find, for *general* β and $k\geqslant l\geqslant 0$,

$$\frac{\Gamma(-\frac{1}{2}k-\beta)\Gamma[\frac{1}{2}(l-k)]}{\Gamma(\beta+1)\Gamma(-\beta)\Gamma(-\frac{1}{2}k)} = (-1)^{(k-l)/2}\frac{(\frac{1}{2}k)!}{\Gamma(\frac{1}{2}k+\beta+1)[\frac{1}{2}(k-l)]!}. \tag{14}$$

Utilising (14) and the duplication formula

$$\Gamma(2x)=2^{2x-1}\pi^{-1/2}\Gamma(x)\Gamma(x+\tfrac{1}{2}) \tag{15}$$

(equation (8.335(1)) in Gradshteyn *et al* (1965)) for the gamma function in (13) we arrive at

$$I(l,\alpha;k,\beta;p)=(-1)^{(k-l)/2}2^{-(\alpha-\beta)}$$

$$\times\frac{k!\Gamma[\frac{1}{2}(\alpha+\beta)-p]\Gamma[\frac{1}{2}(\alpha-\beta)-p]\Gamma[\frac{1}{2}(l+k+1)+\beta]}{\Gamma(\alpha+1)\Gamma(k+1+2\beta)[\frac{1}{2}(k-l)]!\Gamma[\frac{1}{2}(l-k+\alpha-\beta)-p]\Gamma[\frac{1}{2}(l+k+\alpha+\beta+1)-p]}$$

$$\times{}_4F_3\left(\begin{matrix}\frac{1}{2}(\alpha-\beta)+p+1, & \frac{1}{2}(\alpha-\beta)-p, & -\frac{1}{2}(l-1), & -\frac{1}{2}l \\ \frac{1}{2}(k-l)+1, & -\frac{1}{2}(l+k-1)-\beta, & \alpha+1 & \end{matrix};1\right). \tag{16}$$

We make a few remarks about the above formula.

(i) Though it does not manifest the symmetry $l,\alpha\leftrightarrow k,\beta$, it is a very useful formula since the ${}_4F_3$ is well defined for general α and β and $k\geqslant l$ which makes $\frac{1}{2}(k-l)$ a non-negative integer.

(ii) The ${}_4F_3$ is invariant under the transformation $p\leftrightarrow -p-1$. Thus $I(l,\alpha;k,\beta;p)$ and $I(l,\alpha;k,\beta;-p-1)$ are simply related.

(iii) Though we obtained this formula assuming that k and l are both even integers, we obtain the same result when we assume that k and l are both odd integers. For this purpose, we shall use (7*b*) for $P_{l+\alpha}^{-\alpha}(x)$, $P_{k+\beta}^{-\beta}(x)$ in (9) and

$$\int_0^1(1-x^2)^{(\alpha+\beta)/2-p-1+t_1+t_2}x^2\,dx=\tfrac{1}{4}\sqrt{\pi}\,\frac{\Gamma[\frac{1}{2}(\alpha+\beta)-p+t_1+t_2]}{\Gamma[\frac{1}{2}(\alpha+\beta+3)-p+t_1+t_2]} \tag{17}$$

in place of (10) and follow identical steps. Thus the formula is equally valid for both cases.

(iv) For the special case when $\alpha=\beta$ and *integral p*, the hypergeometric function ${}_4F_3$ contains $\min(p+1,|\frac{1}{2}l|+1)$ terms for $p\geqslant 0$ and $\min(-p,|\frac{1}{2}l|+1)$ for $p<0$ where $|\frac{1}{2}l|$ represents the integral part of the non-negative number $\frac{1}{2}l$. In general, the ${}_4F_3$ contains $|\frac{1}{2}l|+1$ terms. We shall make use of this remark in § 4 where we present explicit expressions for $p=1,0,-1,-2$ for the *special integrals.*

3. Evaluation of the special integrals

In order to evaluate the *special integral*

$$\int_{-1}^{1}P_l^m(x)P_k^n(x)(1-x^2)^{-p-1}\,dx$$

for $l\geqslant m\geqslant 0$, $k\geqslant n\geqslant 0$ where l,k,m,n are non-negative integers and $\mathrm{Re}[\frac{1}{2}(m+n)-p]\geqslant 0$, we take α,β in (16) for $I(l,\alpha;k,\beta;p)$ as non-negative integers m and n and replace l,k by the non-negative integers $l-m$, $k-n$. We shall also require

$$P_l^{-m}(x)=(-1)^m\frac{(l-m)!}{(l+m)!}P_l^m(x) \tag{18}$$

(equation (8.752(2)) in Gradshteyn *et al* (1965)) to arrive at

$$\int_{-1}^{1} P_l^m(x)P_k^n(x)(1-x^2)^{-p-1}\,dx$$

$$= (-1)^{(l-k+m-n)/2}2^{-(m-n)}$$

$$\times\frac{(l+m)!\Gamma[\frac{1}{2}(m+n)-p]\Gamma[\frac{1}{2}(m-n)-p]\Gamma[\frac{1}{2}(l+k+1-m+n)]}{(l-m)!m![\frac{1}{2}(k-l+m-n)]!\Gamma[\frac{1}{2}(l-k)-p]\Gamma[\frac{1}{2}(l+k+1)-p]}$$

$$\times{}_4F_3\left(\begin{matrix}\frac{1}{2}(m-n)+p+1, & \frac{1}{2}(m-n)-p, \\ \frac{1}{2}(k-l+m-n)+1, & -\frac{1}{2}(l+k-1-m+n),\end{matrix}\right.$$

$$\times\left.\begin{matrix}-\frac{1}{2}(l-m-1) & -\frac{1}{2}(l-m) \\ m+1 & \end{matrix};1\right) \tag{19}$$

where $k-n \geqslant l-m$. Note that the integral vanishes unless $(k-n)\pm(l-m)$ is an even integer. Note also that in the above result 'p' may not necessarily be an integer. We only require that $\mathrm{Re}[\frac{1}{2}(m+n)-p]>0$ for convergence.

For the special integrals involving Gegenbauer polynomials we first note the relationship

$$(1-x^2)^{\alpha/2-1/4}C_l^\alpha(x)=\sqrt{\pi}\frac{\Gamma(l+2\alpha)}{l!\Gamma(\alpha)}2^{-\alpha+(1/2)}P_{l+\alpha-(1/2)}^{-(\alpha-1/2)}(x) \tag{20}$$

between these polynomials and the associated Legendre functions, valid for a non-negative integer l, $0\leqslant|x|\leqslant 1$ (compare equation (8.932(1)) in Gradshteyn *et al* (1965) and (5) above), which results in

$$\int_{-1}^{1} C_l^\alpha(x)C_k^\beta(x)(1-x^2)^{(\alpha+\beta-3)/2-p}\,dx$$

$$=\pi\frac{\Gamma(l+2\alpha)\Gamma(k+2\beta)}{l!k!\Gamma(\alpha)\Gamma(\beta)}2^{-(\alpha+\beta-1)}I(l,\alpha-\tfrac{1}{2};k,\beta-\tfrac{1}{2};p)$$

$$=(-1)^{(k-l)/2}2^{-(2\alpha-1)}\pi$$

$$\times\frac{\Gamma(l+2\alpha)\Gamma[\frac{1}{2}(\alpha+\beta-1)-p]\Gamma[\frac{1}{2}(\alpha-\beta)-p]}{l!\,\Gamma(\alpha)\Gamma(\beta)\Gamma(\alpha+\frac{1}{2})[\frac{1}{2}(k-l)]!\Gamma[\frac{1}{2}(l-k+\alpha-\beta)-p]}$$

$$\times\frac{\Gamma[\frac{1}{2}(k+l)+\beta]}{\Gamma[\frac{1}{2}(l+k+\alpha+\beta)-p]}$$

$$\times{}_4F_3\left(\begin{matrix}\frac{1}{2}(\alpha-\beta)+p+1, & \frac{1}{2}(\alpha-\beta)-p & -\frac{1}{2}(l-1), & -\frac{1}{2}l \\ \frac{1}{2}(k-l)+1, & -\frac{1}{2}(l+k)-\beta+1, & \alpha+\frac{1}{2} & \end{matrix};1\right) \tag{21}$$

where k and l are either both even or both odd non-negative integers, $k\geqslant l$ and $\mathrm{Re}[\frac{1}{2}(\alpha+\beta-1)-p]>0$. When one of l and k is even and the other an odd non-negative integer, the integral is zero.

4. Special integrals for the case when $m=n(\alpha=\beta)$ and p an integer

Defining

$$J(l,k,m;p)=\int_{-1}^{1} P_l^m(x)P_k^m(x)(1-x^2)^{-p-1}\,dx$$

and

$$K(l, k, \alpha; p) = \int_{-1}^{1} C_l^\alpha(x) C_k^\alpha(x)(1-x^2)^{\alpha-(3/2)-p}\, dx$$

we use (19) and (21) to find

$$J(l, k, m; 0) = \frac{1}{m} \frac{(l+m)!}{(l-m)!} \qquad \text{where } k \geqslant l \geqslant m > 0$$

$$J(l, k, m; 1) = \frac{(l+m)!}{2(l-m)!(m-1)m(m+1)}$$
$$\times [l(l+1) + (m-1)(m+1) + \tfrac{1}{2}(k-l)(m+1)(k+l+1)]$$

for $k \geqslant l \geqslant m > 1$

$$J(l, k, m; -1) = \frac{2}{(2l+1)} \frac{(l+m)!}{(l-m)!} \delta_{kl}$$

$$J(l, k, m; -2)$$
$$= (-1)^{(k-l)/2} \frac{(l+m)![l(l+1) + (m-1)(m+1) + \frac{1}{2}(k-l)(m+1)(k+l+1)]}{(l-m)![\frac{1}{2}(k-l)+1]!\Gamma[\frac{1}{2}(l-k)+2](k+l-1)(k+l+1)(k+l+3)}$$

for $k \geqslant l \geqslant m \geqslant 0$. Note that $J(l, k, m; -2)$ is zero when $k-l \geqslant 4$.

$$K(l, k, \alpha; 0) = \pi 2^{-(2\alpha-1)} \frac{\Gamma(l+2\alpha)}{(\alpha-\frac{1}{2})[\Gamma(\alpha)]^2} \qquad \text{for } k \geqslant l \geqslant 0,\ \alpha > \tfrac{1}{2}$$

$$K(l, k, \alpha; 1) = \pi 2^{-2\alpha} \frac{\Gamma(l+2\alpha)}{(\alpha-\frac{3}{2})(\alpha-\frac{1}{2})(\alpha+\frac{1}{2})[\Gamma(\alpha)]^2}$$
$$\times [(l+\alpha-\tfrac{1}{2})(l+\alpha+\tfrac{1}{2}) + (\alpha-\tfrac{3}{2})(\alpha+\tfrac{1}{2}) + \tfrac{1}{2}(k-l)(\alpha+\tfrac{1}{2})(k-l+2\alpha)]$$

for $k \geqslant l \geqslant 0,\ \alpha > \frac{3}{2}$

$$K(l, k, \alpha; -1) = \pi 2^{-(2\alpha-1)} \frac{\Gamma(l+2\alpha)}{l!\,(l+\alpha)[\Gamma(\alpha)]^2} \delta_{kl} \qquad \text{where } \alpha > -\tfrac{1}{2}$$

$$K(l, k, \alpha; -2) = (-1)^{(k-l)/2} \pi 2^{-2\alpha} \frac{\Gamma(l+2\alpha)}{l![\Gamma(\alpha)]^2}$$
$$\times \left(\frac{(l+\alpha-\frac{1}{2})(l+\alpha+\frac{1}{2}) + (\alpha-\frac{3}{2})(\alpha+\frac{1}{2}) + \frac{1}{2}(k-l)(\alpha+\frac{1}{2})(k+l+2\alpha)}{[\frac{1}{2}(k-l)+1]!\Gamma[\frac{1}{2}(l-k)+2][\frac{1}{2}(k+l)+\alpha-1][\frac{1}{2}(k+l)+\alpha][\frac{1}{2}(k+l)+\alpha+1]} \right)$$

for $k \geqslant l \geqslant 0,\ \alpha > -\frac{3}{2}$.

Note that $K(l, k, \alpha; -2)$ is zero for $|k-l| \geqslant 4$.

All the above integrals have been evaluated with $k \geqslant l$ which can always be arranged on account of the symmetry of the special integrals. The results in this section are generalisations to $k \geqslant l$ of the results obtained earlier by Laursen and Mita (1981) for $k = l$.

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